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On the spatial Fourier transforms of the Pascal–Sierpinski gaskets

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Abstract. It has been shown here that the Fourier transform, which can be obtained experimentally, of an evolving fractal can be manipulated to yield the fractal dimension. Although only the Pascal-Sierpinski gaskets have been considered here, because of their generality, it is expected that this technique can be utilised to identify other planar fractals as well.

1. Introduction

In a recent communication (Holter *et al* 1986, hereafter referred to as I), we have described a new class of planar fractals called the Pascal-Sierpinski gaskets (PSG) of integral orders N. We have shown that the PSG of prime orders are strictly self-similar, and for which a fractal (similarity) dimension $d_N = \log\{1+2+\ldots+N\}/\log\{N\}$ can be prescribed. PSG whose order N is any integral power of a prime are self-affine, whereas there is no visually discernible scaling relationship when N is neither a prime nor an integral order of a prime. Nevertheless, in I we have shown that, regardless of the specific properties of the order N, all PSG can be characterised by non-integral or fractal dimensions. Consequently, the PSG form a broad family of fractals and may be considered representative of ordered planar fractal structures.

We shall not go into the possible applications of gaskets here, for which purpose a cursory glance at the contents of a few recent issues of this journal would suffice. Instead, we are definitely interested in the possibility of recovering the fractal dimension of a PSG from its spatial Fourier transform (SFT). SFT can nowadays be taken optically quite easily (Goodman 1968) and the recovery of the fractal dimension from them would greatly enlarge our ability in identifying fractals. Since the PSG form a fairly representative family of fractals, the conclusions drawn from studying their SFT on this score could apply to planar fractals in general.

2. Preliminaries

Consider, therefore, the triangular grid shown in figure 1, in which the nth row ($n = 0, 1, 2, \ldots$) contains $n+1$ number of sites $\{n, m_n\}$, $m_n = 0, 1, 2, \ldots, n$. A cartesian

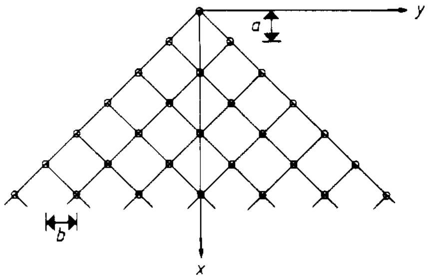

Figure 1. The triangular grid over which the Pascal–Sierpinski gaskets evolve. The cartesian coordinate system is also shown.

coordinate system is also imposed on this grid, as also shown in figure 1, so that consecutive rows (parallel to the y axis) are spaced a distance a apart, while the adjacent sites along each row are spaced $2b$ apart, the ratio $a/b \geq 1$ being constant for a given family of PSG. The site $\{n, m_n\}$ is considered empty if the binomial coefficient $\{n![m_n!(n-m_n)!]^{-1}\}$ is exactly divisible by an integer $N > 1$; otherwise, that site is occupied by a unit point mass. These occupied sites form a PSG of order N, with $N = 2$ corresponding to the usual Sierpinski gasket (Mandelbrot 1983) and $N = 3$ to a gasket described by Bhattacharya (1985). Needless to add, each of the PSG has to be truncated after so many rows.

Each PSG of order N can be mathematically described in terms of a function $f(x, y; N)$ to be given as

$$\begin{aligned} f(x, y; N) &= 1 \qquad \text{if } (x, y) \text{ is an occupied site} \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{1}$$

The SFT of this function, in terms of spatial frequencies u and v, is defined as the double integral

$$F(u, v; N) = \int \mathrm{d}x \int \mathrm{d}y \exp[-\mathrm{i}(ux + vy)] f(x, y; N) \tag{2}$$

implemented over the domain $-\infty \leq x, y \leq \infty$. Now, if $f(x, y; N)$ is a fractal, it follows that (Ausloos and Berman 1985)

$$f(x/\alpha, y/\alpha; N) = \alpha^{P(d_N)} f(x, y; N) \qquad \alpha > 0 \tag{3}$$

where $P(\ )$ is some linear function and $1 < d_N \leq 2$. As a result of (2) and (3), it should be expected that $F(u, v; N)$ should also exhibit the fractal dimension. This expectation forms the cornerstone of this paper.

3. PSG of prime order

Let us consider, first, a PSG whose order N is a prime. Such a PSG is self-similar with a scale factor also of N. As such, it makes sense for us to truncate the PSG when the number of rows is an integral power of N, i.e. we define *levels of evolution* $L \geq 1$, each level containing the first N^{L+1} rows. The fundamental level is when $L = 1$ and is shown graphically in figure 2 for $N = 2$, 3 and 5.

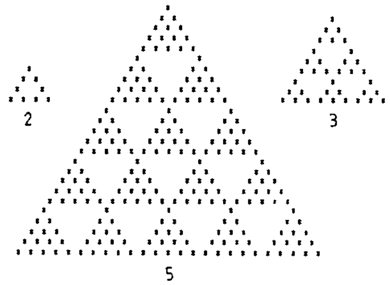

Figure 2. The fundamental level of evolution, $L=1$, for the PSG of orders 2, 3 and 5.

We also define a mathematical representation for each level as

$$f_L(x, y; N) = f(x, y; N) \qquad \forall n \leqslant N^{L+1}$$
$$= 0 \qquad \text{otherwise} \tag{4}$$

and observe that

$$f_L(x, y; N) = f_{L-1}(x, y; N) * g_L(x, y; N) \tag{5a}$$

where $*$ denotes a convolution (Goodman 1968), the array factor

$$\begin{aligned} g_L(x, y; N) = & \sum_{p=3,5,\ldots}^{N} \sum_{q=2,4,\ldots}^{p-1} [\delta\{x-(p-1)N^L a, y - qN^L b\} \\ & + \delta\{x-(p-1)N^L a, y+qN^L b\}] + \sum_{p=1,3,5,\ldots}^{N} [\delta\{x-(p-1)N^L a, y\}] \\ & + \sum_{p=2,4,6,\ldots}^{N} \sum_{q=1,3,5,\ldots}^{p-1} [\delta\{x-(p-1)N^L a, y-qN^L b\} \\ & + \delta\{x-(p-1)N^L a, y+qN^L b\}] \end{aligned} \tag{5b}$$

and $\delta\{x-x', y-y'\}$ is the Dirac delta function. At the same time, the SFT of these levels come out to be related simply as

$$F_L(u, v; N) = F_{L-1}(u, v; N) G_L(u, v; N) \tag{6a}$$

where

$$\begin{aligned} G_L(u, v; N) = & \sum_{p=1,3,5,\ldots}^{N} \sum_{q=0,2,4,\ldots}^{p-1} \{[2/(1+\delta_{q0})] \exp[-\mathrm{i}(p-1)N^L au] \cos(qN^L bv)\} \\ & + \sum_{p=2,4,6,\ldots}^{N} \sum_{q=1,3,5,\ldots}^{p-1} \{2 \exp[-\mathrm{i}(p-1)N^L au] \cos(qN^L bv)\} \end{aligned} \tag{6b}$$

and $\delta_{qq'}$ is the Kronecker delta. It should be noted that the identity

$$G_L(u, v; N) = G_{L-1}(Nu, Nv; N) \tag{6c}$$

devolves from the definition (5b).

In order to obtain a fractal dimension, let us consider a triangular area which is congruently filled by a PSG of order N, evolution L and inter-site spacings a and b. It can thus be completely specified by the function $f_L(x, y; N)$ whose SFT is $F_L(u, v; N)$. Let it now evolve to the next level $L+1$ while still remaining congruent with the triangular area. Consequently, this higher level structure has inter-site spacings a/N

and b/N, and it can be represented by $f_{L+1}(x/N, y/N; N)$ whose SFT is $N^2 F_{L+1}(Nu, Nv; N)$ due to the scaling properties of the Fourier transforms. Using (5a), it can be deduced that the ratio

$$\begin{aligned}R_{L/L+1}(u, v; N) &\\ &= F_L(u, v; N)/N^2 F_{L+1}(Nu, Nv; N)\\ &= [F_1(u, v; N)/N^2 F_1(Nu, Nv; N)]\\ &\quad\times [G_2(u, v; N)/G_{L+1}(u, v; N) G_{L+2}(u, v; N)]. \end{aligned}\tag{7}$$

If this ratio is evaluated at the spatial frequencies u and v such that both au and bv are multiples of 2π, then it is positive real and its value is given by

$$N^{-2-\log\{1+2+\ldots+N\}/\log\{N\}}\tag{8}$$

However, $\log\{1+2+\ldots+N\}/\log\{N\} = d_N$, the similarity dimension for these PSG. Hence it can be concluded that the fractal dimension of the PSG of prime order is recoverable from their SFT at any two successive evolutionary levels. Shown in figure 3 is the SFT of the PSG of order $N=2$ when it has evolved so as to include eight rows, i.e. $L=2$.

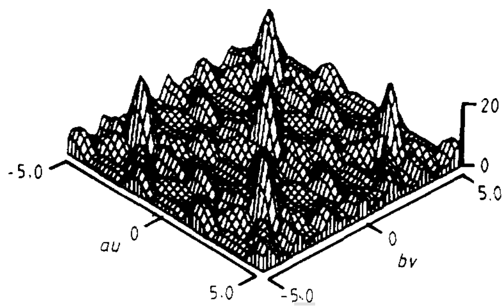

Figure 3. The Fourier transform of a Sierpinski gasket ($N=2$) which contains eight rows ($L=2$). The inter-row spacing $a=1$ while the semi-inter-site spacing $b=\tan(\pi/6)$.

4. PSG of non-prime order

For a PSG whose order is not a prime, there are neither clear-cut levels of evolution, nor a scale factor. Hence, the method of the previous section does not apply. Let us, however, arbitrarily assign levels of evolution $L \geq 1$ such that each level contains the first NL rows.

Let, initially, a triangular area be congruently occupied by a PSG of order N, level L and inter-site spacings a and b. Then let it successively evolve through levels $L+1, L+2, \ldots$ till it reaches a level L' whose inter-site spacings are a/s' and b/s', the factor $s' = L'/L$. It is to be expected that the SFT of these two levels L and L' should be related by a ratio which could yield a relationship like (8) at some frequencies u and v. Very specifically, we expect that the ratio

$$R_{L/L'}(0, 0; N) = F_L(0, 0; N)/s'^2 F_{L'}(0, 0; N) \approx s'^{\{-2-D(L,L')\}}\tag{9}$$

and the mean of the numbers $D(L, L')$ determined over several levels should give the fractal dimension d_N. This expectation comes from the fact that the SFT of any function, when evaluated at zero frequencies, is the zeroth moment of that function.

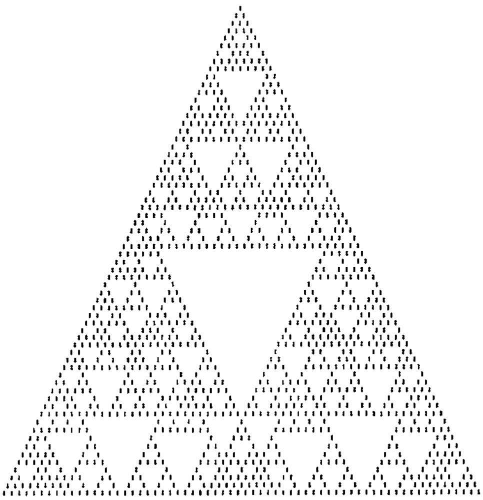

Figure 4. The PSG of order $N = 6$. The first 64 rows are shown here.

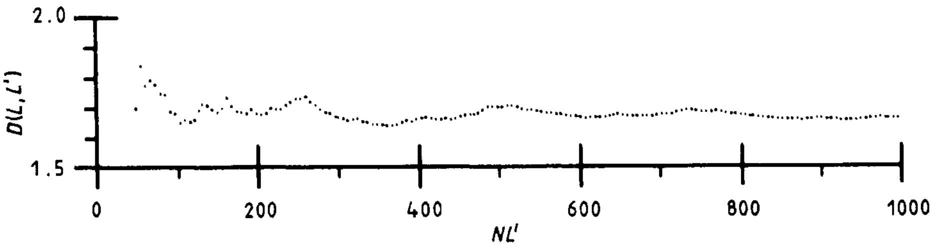

Figure 5. Showing the computed function $D(L, L')$ as a function of NL' for a PSG of order 6. The initial level is set to be $L = 6$, while $L' \geqslant 166$.

Shown in figure 4 is the PSG of order $N = 6$, for which, in figure 5, the quantity $D(L, L')$ is plotted against L', with $L = 6$ being the initial level. The mean of this quantity over the range $48 \leqslant L' \leqslant 166$ is determined to be 1.670 103 which must be a close approximation to the fractal dimension d_6 of the PSG of order 6, because the mass-radius dimension of the first 1000 rows of the order 6 PSG was computed in I to be 1.6693. This closeness between d_6 computed here and in I points out the premise that the determinations, using different methods, of the fractal dimension of non-self-similar fractals are approximate and hover around a 'true' fractal dimension. Indeed, as has been pointed out by Stanley (1986), several fractal dimensions can be thought of for a non-self-similar fractal, all of which may not be independent of each other. Thus, further exploration of the ratio $R_{L/L'}(u, v; N)$ for non-zero u and v may show that the SFT of fractals may be holding even more promise, and this is an area which requires further investigation. In any case, the adequacy of using SFT to determine fractal dimensions has been further reinforced here.

5. Conclusion

It has been shown here that the Fourier transform, which can be obtained experimentally, of an evolving fractal can be manipulated to yield the fractal dimension.

Although only the PSG have been considered here, because of their generality, it is expected that this technique can be utilised to identify other planar fractals as well.

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